Digital Signal Processing Fundamentals

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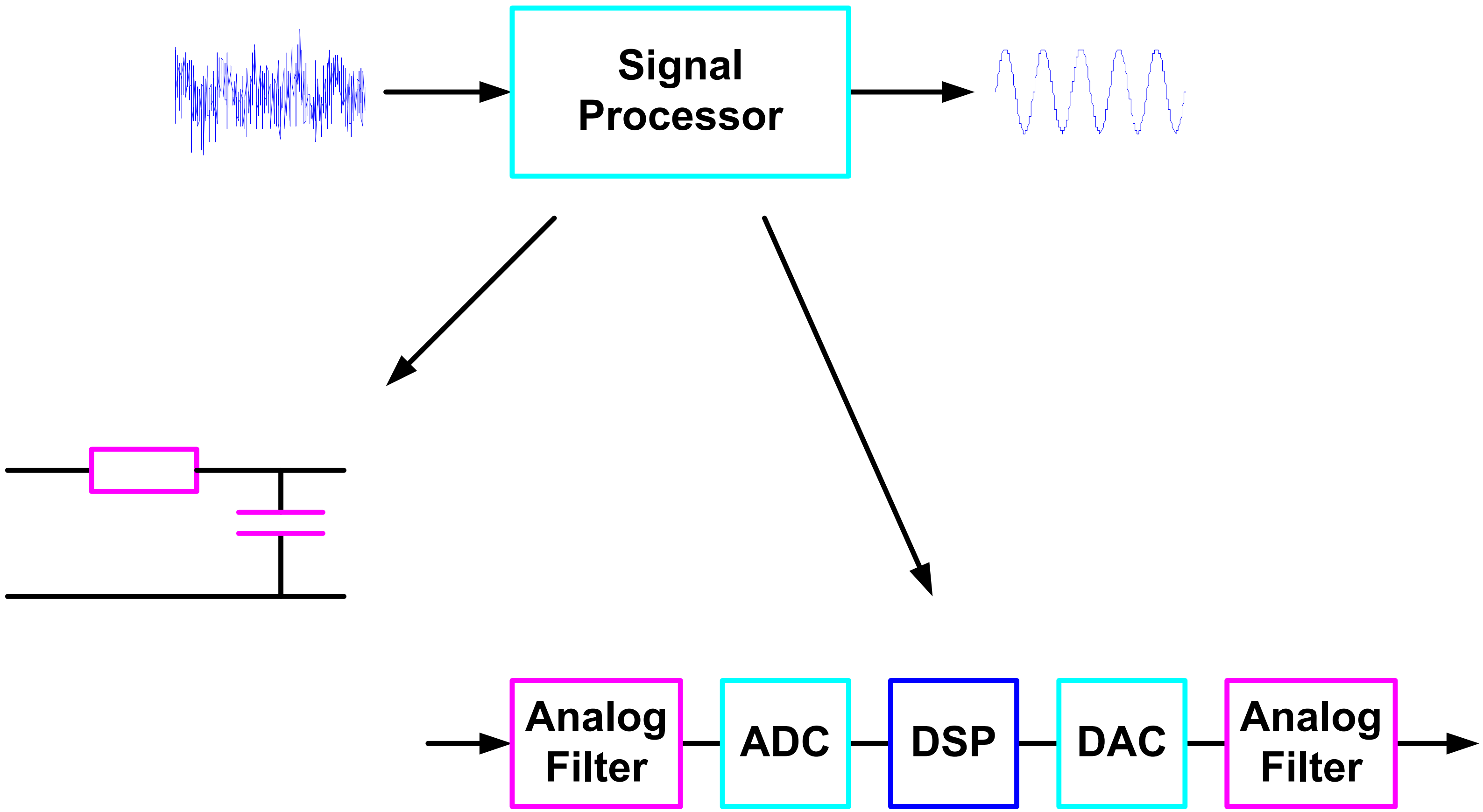
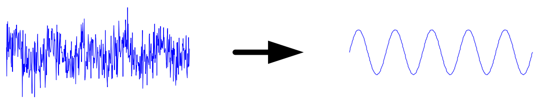
“Black Box” View

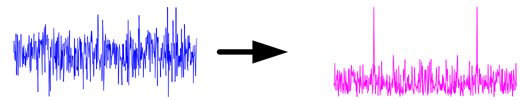


Key DSP Operations

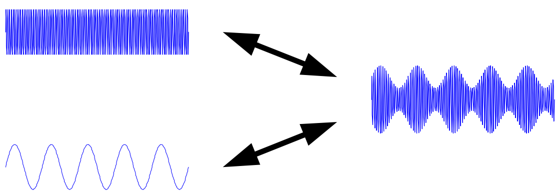
- Filtering



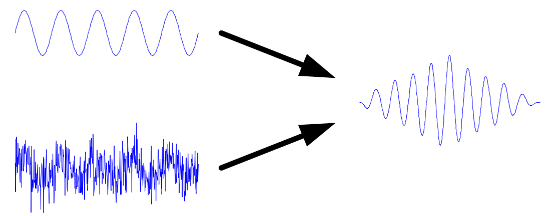
- Transformation into another domain



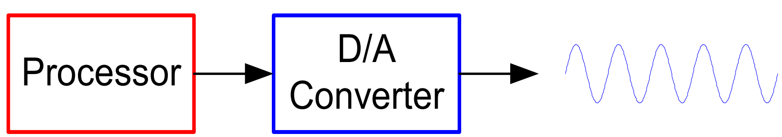
- Modulation and demodulation



- Correlation of two signals

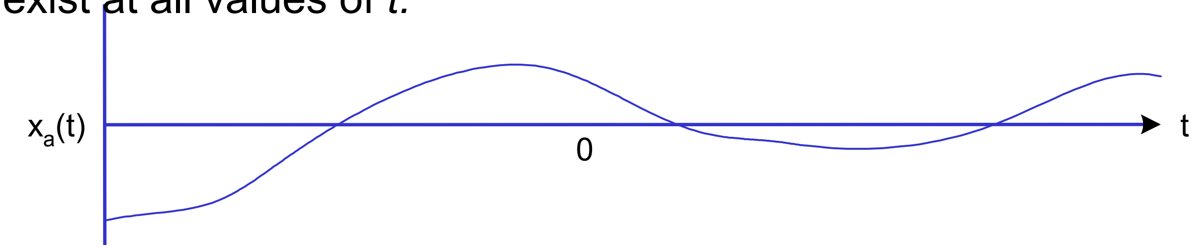


- Signal generation, frequency synthesis

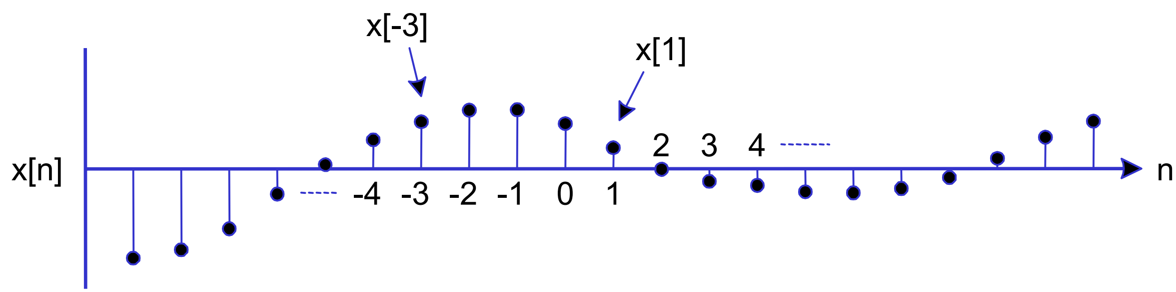


Discrete-Time vs Continuous-Time

- Continuous-time signals are functions of a continuous-valued independent variable *t*.
- They exist at all values of *t.*



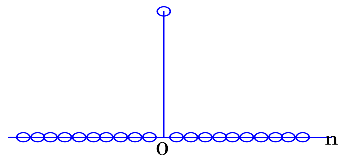
- By contrast, discrete-time signals are functions of an integer-valued index (eg *n, m, k*)
- The signals have no meaning for non-integer values of the independent variable.



- We will try to follow the convention of representing a continuous-time signal with (), eg *x(t)*, and discrete-time signals with [], eg x*[k]*.

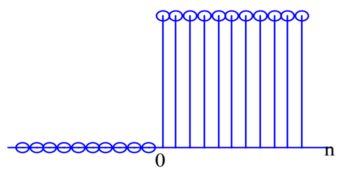
Elementary Discrete-Time Signals

Unit impulse



$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Unit step



$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

The unit step and unit impulse are related as follows

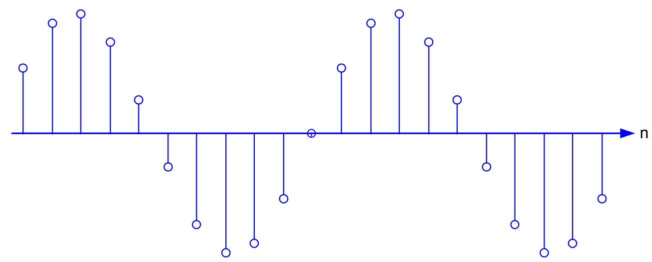
$$u[n] = \sum_{k=0}^{n} \delta(n-k) \qquad \delta[n] = u[n] - u[n-1]$$

Less-Elementary Discrete-Time Signals

Sinusoid

$$x[n] = \sin(\omega_o n + \phi)$$

Note that ω_o must be a rational multiple of π for x[n] *to be periodic.*

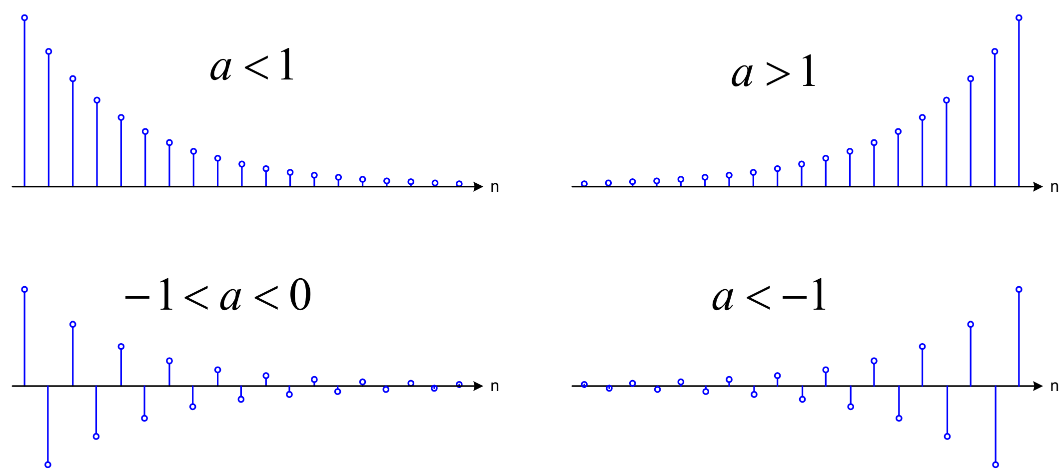


Real Exponential

$$x[n] = a^n$$

C-T equivalent:

$$x(t) = e^{-at}$$

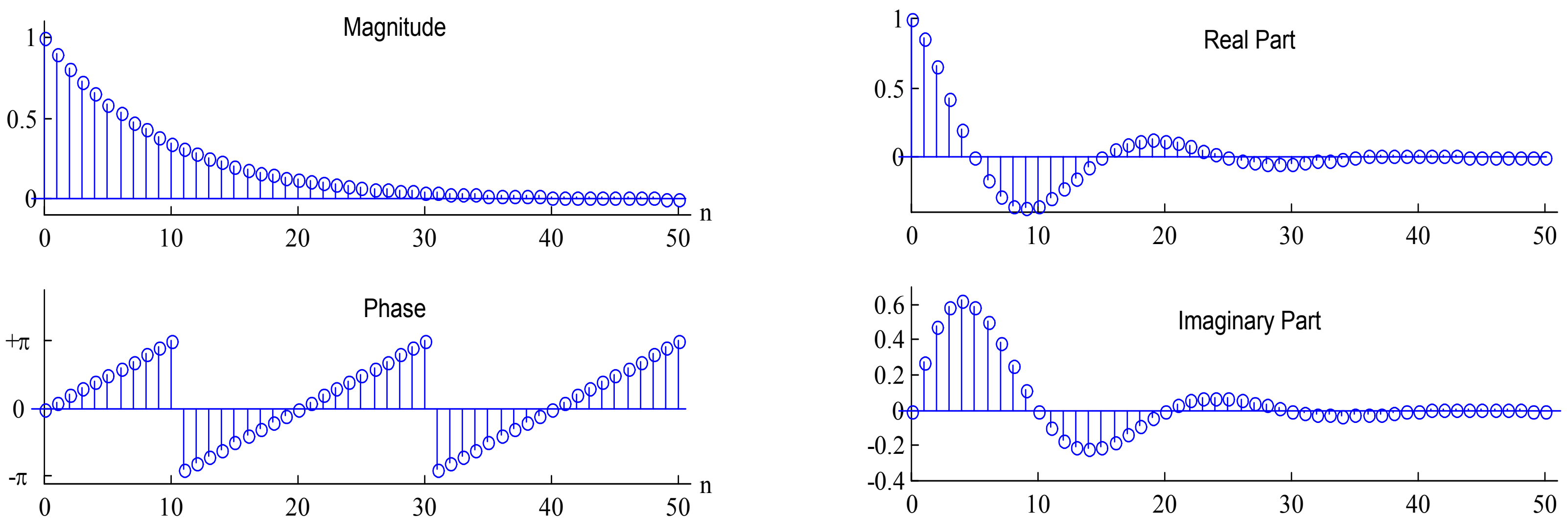


Less-Elementary Discrete-Time Signals (cont)

Complex Exponential

$$x[n] = a^n \qquad \text{where} \qquad a = e^{j\phi} \cdot e^{(\sigma_o + j\omega_o)} \qquad \begin{aligned} \phi &= phase \\ \omega &= frequency \\ \sigma &= damping \end{aligned}$$

ie $$x[n] = e^{j\phi} \cdot e^{(\sigma_o + j\omega_o)n} = e^{\sigma_o n} \cdot [\cos(\omega_o n + \phi) + j \sin(\omega_o n + \phi)]$$



Discrete-Time Frequency Units

- Consider the continuous-time sinusoid with continuous-time frequency f_c

$$x_a(t) = A \cdot \cos(2\pi f_c t) = A \cdot \cos(\Omega_c t)$$

- Sampling this at intervals T $(=1/F_s)$ results in the discrete-time sequence

$$\begin{aligned} x[n] &= A \cdot \cos(\Omega_c T n) \\ &= A \cdot \cos\left(\frac{\Omega_c}{F_s} \cdot n\right) = A \cdot \cos\left(\frac{2\pi \cdot \Omega_c}{\Omega_s} \cdot n\right) \\ &= A \cdot \cos(\omega_d \cdot n) \end{aligned}$$

where,

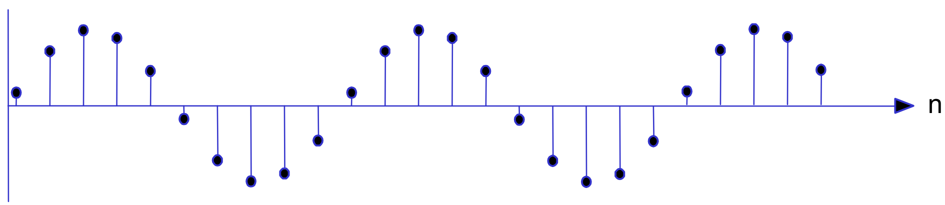
$$nT = \frac{n}{F_s} = \frac{2\pi n}{\Omega_s} \qquad \text{and} \qquad \omega_d = \frac{2\pi \cdot \Omega_c}{\Omega_s} = \Omega_c T$$

- The units of the discrete-time frequency ω_d are radians per sample, or simple radians, with range

$$-\pi < \omega_d < \pi \qquad \text{or} \qquad 0 < \omega_d < 2\pi$$

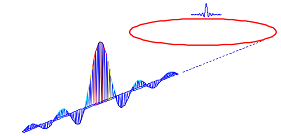
Discrete-Time Frequency Units (cont)

- How would we determine the continuous-time frequency associated with the discrete-time sequence below?



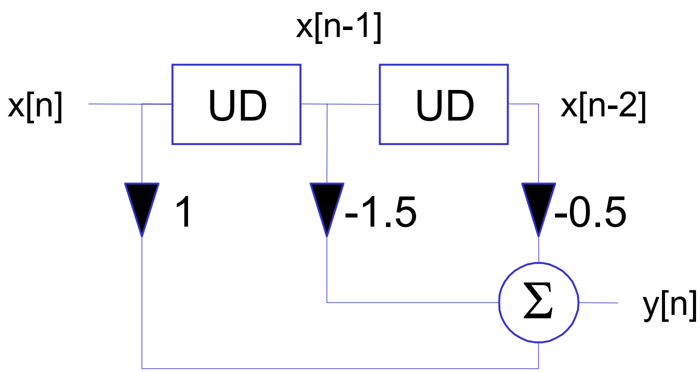
- If we are not given the sampling frequency, we can do no better than to determine the frequency in terms of cycles/sample. In this example, each sinusoid has 10 samples, so the discrete-time frequency is 0.1 cycles/sample.
- Examples

Continuous-time		Sampling frequency	Discrete-time	
Cycles/sec	Radians/sec		Cycles	Radians
100Hz	200π	1000Hz	0.1	$\pi/5$
1Hz	2π	2Hz	0.5	π



Representing Discrete-Time Systems

- Consider the following block diagram



- The output *y[n]* difference equation is

$$y[n] = x[n] - 1.5x[n-1] - 0.5x[n-2]$$

- To compute the impulse response, we make *x[n]* a delta function, ie

$$x[0] = 1$$
$$x[1] = 0$$
$$\vdots$$

So,

$$h[0] = x[0] - 1.5x[-1] - 0.5x[-2] = 1$$
$$h[1] = x[1] - 1.5x[0] - 0.5x[-1] = 1.5$$
$$h[2] = x[2] - 1.5x[1] - 0.5x[0] = -0.5$$
$$h[3] = x[3] - 1.5x[2] - 0.5x[1] = 0$$
$$\vdots$$

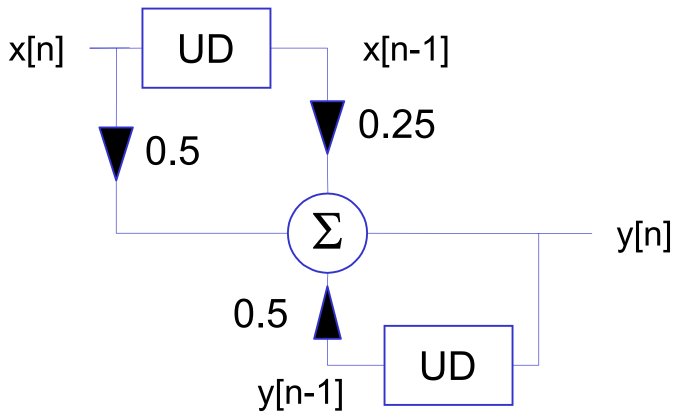
- The impulse response sequence is therefore

$$h[n] = \{1, -1.5, -0.5\}$$

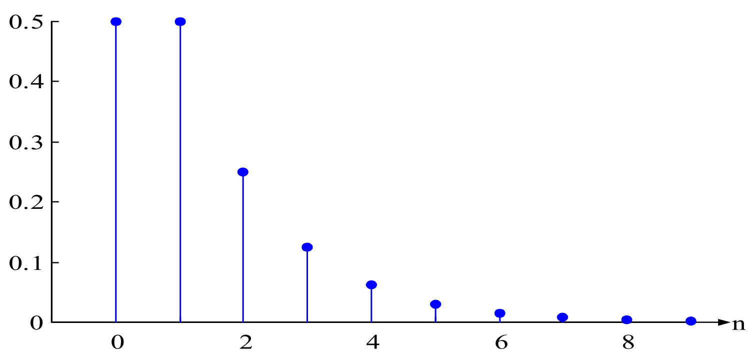
- This is a Finite Impulse Response (FIR) system.

Recursive Systems

- The following system is recursive (it uses past output values in the computation of the present output value).



First nine points of the impulse response



- The difference equation is

$$y[n] = 0.5x[n] + 0.25x[n-1] + 0.5y[n-1]$$

- The impulse response is

$$\begin{aligned} h[0] &= 0.5\delta[0] + 0.25\delta[-1] + 0.5y[-1] \\ &= 0.5 \\ h[1] &= 0.5\delta[1] + 0.25\delta[0] + 0.5y[0] \\ &= 0.5 \\ h[2] &= 0.5\delta[2] + 0.25\delta[1] + 0.5y[1] \\ &= 0.25 \\ &\vdots \\ h[k] &= 0.5y[k-1] \end{aligned}$$

- This is an Infinite Impulse Response (IIR) system.

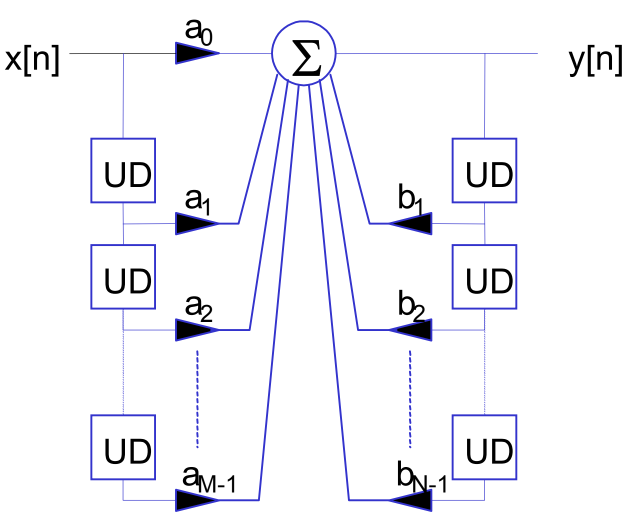
General Description of LTI Systems

- All LTI systems can always be represented by constant-coefficient difference equations of the form

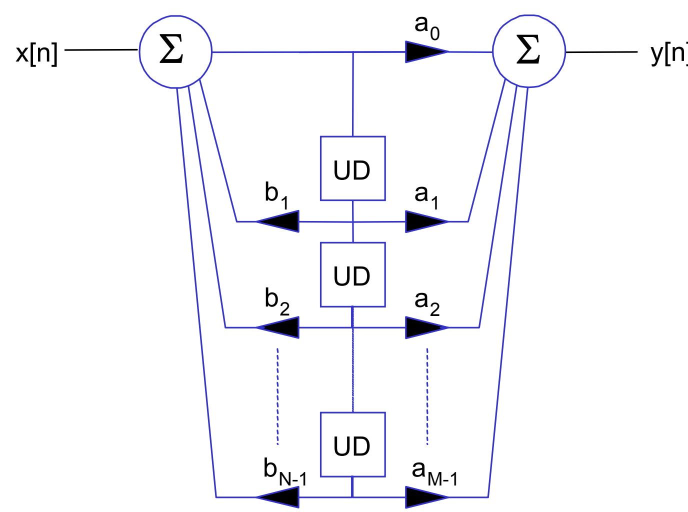
$$y[n] = \sum_{k=0}^{M-1} a_k x[n-k] - \sum_{k=1}^{N-1} b_k y[n-k]$$

- The two common realizations of this general difference equation are

Direct-Form

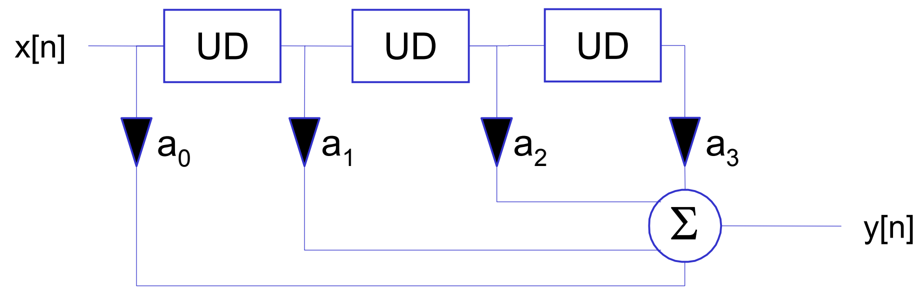


Canonical Form



Discrete-Time Convolution

- Consider the system described by the following block diagram



- The difference equation is

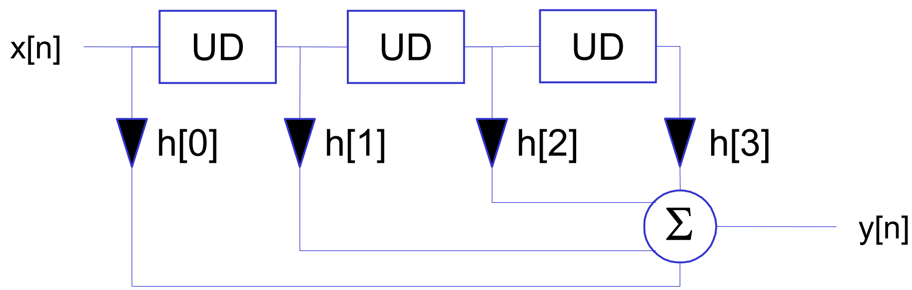
$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] + a_3 x[n-3]$$

- And from previous discussion, we can deduce that the impulse response sequence is

$$h[n] = \{a_0, a_1, a_2, a_3\}$$

Discrete-Time Convolution (cont)

- Since in this case, the impulse response sequence and the system coefficients are one and the same, we can replace the coefficients by the impulse response sequence



- The difference equation is now given by

$$y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + h[3]x[n-3]$$

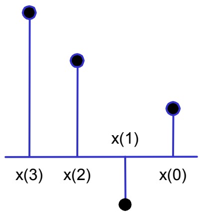
Or equivalently

$$y[n] = \sum_{k=0}^{3} h[k]x[n-k]$$

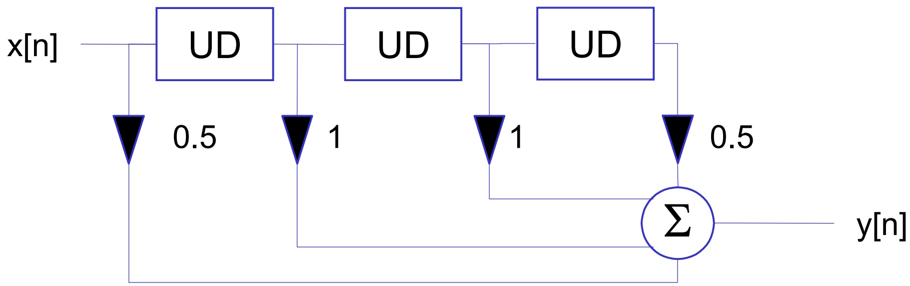
- This is the convolution equation for a general 4-coefficient FIR system

Discrete-Time Convolution Example

- Convolve the following input and impulse response sequences

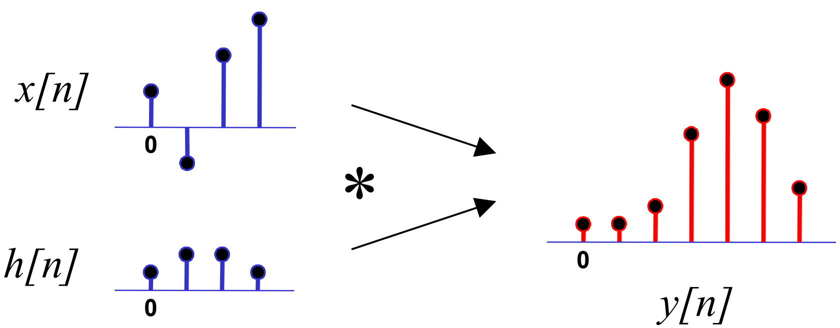


$$x[n] = \{1, -1, 2, 3, 0, 0, \ldots\} \qquad h[n] = \{0.5, 1, 1, 0.5\}$$



$$y[n] = \sum_{k=0}^{3} h[k]x[n-k]$$

Computing the output point-by-point, we get



$$y[0] = 0.5(1) = 0.5$$
$$y[1] = 0.5(-1) + 1(1) = 0.5$$
$$y[2] = 0.5(2) + 1(-1) + 1(1) = 1$$
$$y[3] = 0.5(3) + 1(2) + 1(-1) + 0.5(1) = 3$$
$$y[4] = 0.5(0) + 1(3) + 1(2) + 0.5(-1) = 4.5$$
$$y[5] = 0.5(0) + 1(0) + 1(3) + 0.5(2) = 4$$
$$y[6] = 0.5(0) + 1(0) + 1(0) + 0.5(3) = 1.5$$

IIR Convolution Example

$$h[0] = (0.9)^0 = 1$$
$$h[1] = (0.9)^1 = 0.9$$
$$h[2] = (0.9)^2 = 0.81$$
$$\vdots$$

- Convolve the following signal and causal impulse response

$$x[n] = \{2,3,1,0,\ldots\} \qquad h[n] = (0.9)^n$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} (0.9)^k x[n-k]$$

- Now *x[n]* is non-zero only for $0 \le n \le 2$, and *h[n]* is non-zero only when $n \ge 0$, so we only need consider the cases when

$$0 \le (n-k) \le 2 \qquad \text{and} \qquad k \ge 0$$

- For *n=0*, only *k=0* is valid, so we get

$$y[0] = (0.9)^0 x[0] = (1)x[0] = 2$$

- For *n=1*, both *k=0*, and *k=1* satisfy the limits, so

$$y[1] = (0.9)^0 x(1) + (0.9)^1 x[0] = (1)(3) + (0.9)(2) = 4.8$$

IIR Convolution Example (cont)

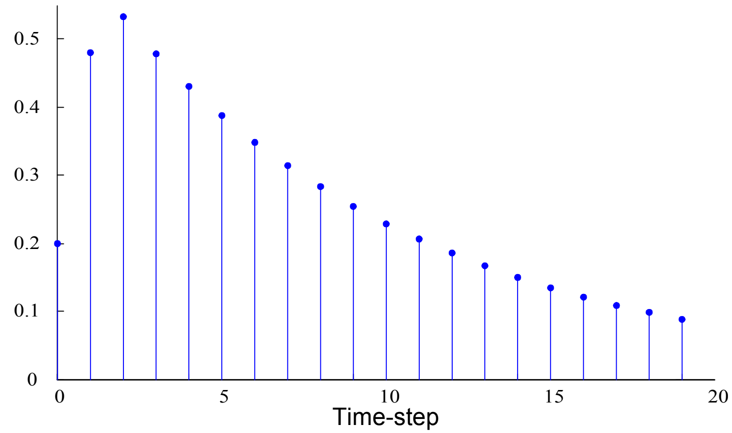
- For *n=2*, both *k=0*, *k=1* and *k=2* are valid, so

$$y[2] = (0.9)^0 x[2] + (0.9)^1 x[1] + (0.9)^2 x[0] = (1)(1) + (0.9)(3) + (0.81)(2) = 5.32$$

- For *n=3*, we find that *k=1*, *k=2* and *k=3* are valid, so

$$y[3] = (0.9)^1 x[2] + (0.9)^2 x[1] + (0.9)^3 x[0] = (0.9)(1) + (0.81)(3) + (0.729)(2) = 4.788$$

- The first 20 points are plotted below



Z-Transforms

- The Z-transform is the discrete-time equivalent of the Laplace Transform.

Definition

$$X(z) := \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$

- The $\boldsymbol{z^{-1}}$ operator is the equivalent of a unit (one-sample) delay.

Example

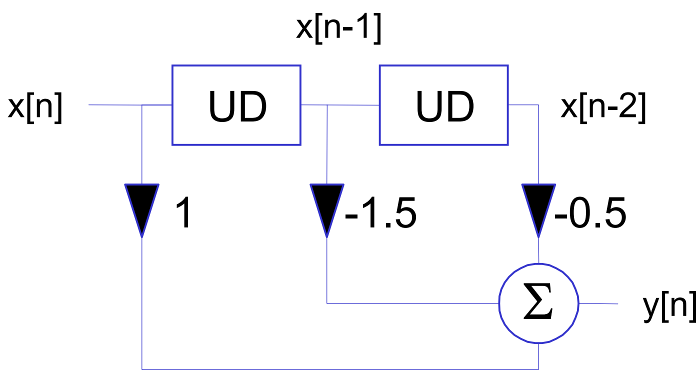
$$x[k] = \left(\frac{1}{2}\right)^{k} u[k]$$

$$\Rightarrow X(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} z^{-k}$$

$$\Rightarrow X(z) = \frac{1}{1-0.5z^{-1}} \quad , \quad |z| > 0.5 \qquad \text{(using Taylor series)}$$

Transfer functions in the Z-domain

- FIR example…

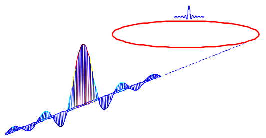


Difference equation

$$y[n] = x[n] - 1.5x[n-1] - 0.5x[n-2]$$

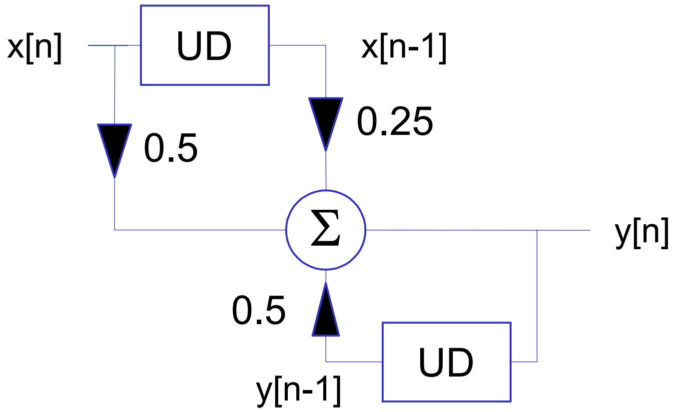
Z-transform of d.e. $$Y(z) = X(z) - 1.5z^{-1}X(z) - 0.5z^{-2}X(z)$$

Transfer function $$\frac{Y(z)}{X(z)} = 1 - 1.5z^{-1} - 0.5z^{-2}$$



Transfer functions in the Z-domain

- IIR example…



Difference equation

$$y[n] = 0.5x[n] + 0.25x[n-1] + 0.5y[n-1]$$

Z-transform of d.e. $\quad Y(z) = 0.5X(z) + 0.25z^{-1}X(z) + 0.5z^{-1}Y(z)$

Transfer function $\quad \dfrac{Y(z)}{X(z)} = \dfrac{0.5 + 0.25z^{-1}}{1 - 0.5z^{-1}}$

Some common Z-transform pairs

Sequence	Z-transform	ROC
δ[k]	1	All values of z
u[k]	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$a^k u[k]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$(r^k \cos\omega_0 k)u[k]$	$\frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > r$
$(r^k \sin\omega_0 k)u[k]$	$\frac{(r\sin\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Inverse z-transforms

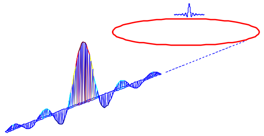
- Use partial fraction expansions to generate sum of terms that can be inverted by inspection

Example

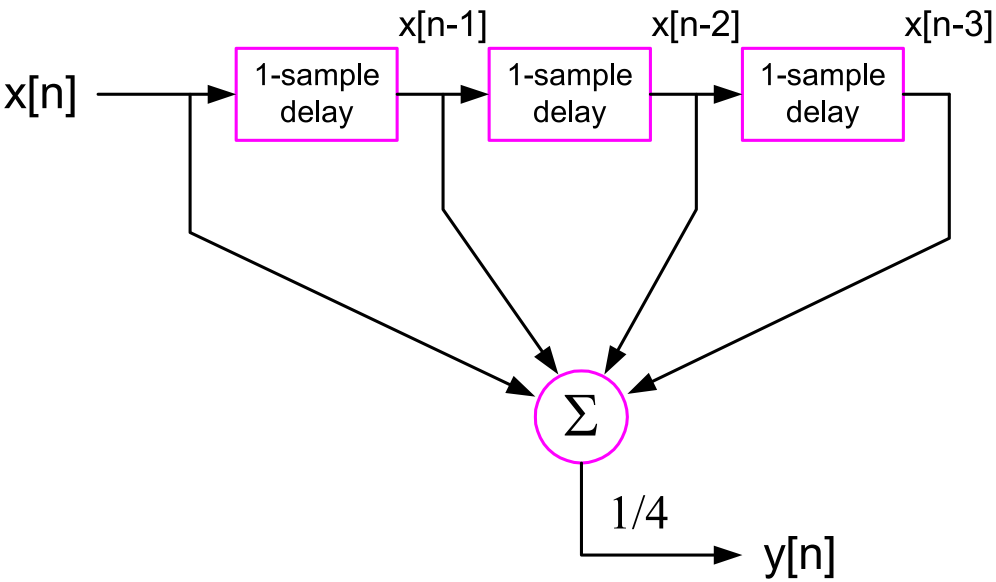
$$X(z)=\frac{1-\frac{1}{3}z^{-1}}{1+\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}}$$

$$X(z)=\frac{1-\frac{1}{3}z^{-1}}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-2}\right)}=\frac{2/3}{1+\frac{1}{3}z^{-1}}+\frac{-1}{1-\frac{1}{2}z^{-1}}$$

$$\therefore x[k]=\frac{2}{3}\left(-\frac{1}{3}\right)^{k}u[k]-\left(\frac{1}{2}\right)^{k}u[k]$$



Averager Block Diagram (DSP Viewpoint)



- This can be described with the following *difference equation*

$$y[n] = 0.25 \cdot \left(x[n] + x[n-1] + x[n-2] + x[n-3]\right)$$

- Or with the following z-transform transfer function

$$H_{lp}(z) = \frac{Y(z)}{X(z)} = \frac{1}{4}(1 + z^{-1} + z^{-2} + z^{-3})$$

Frequency response

- Evaluate frequency response by setting

$$z = e^{jw}$$

Then the frequency response of the 4-point averager is:

$$H_{lp}(e^{j\omega}) = \frac{1}{4}(1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega})$$

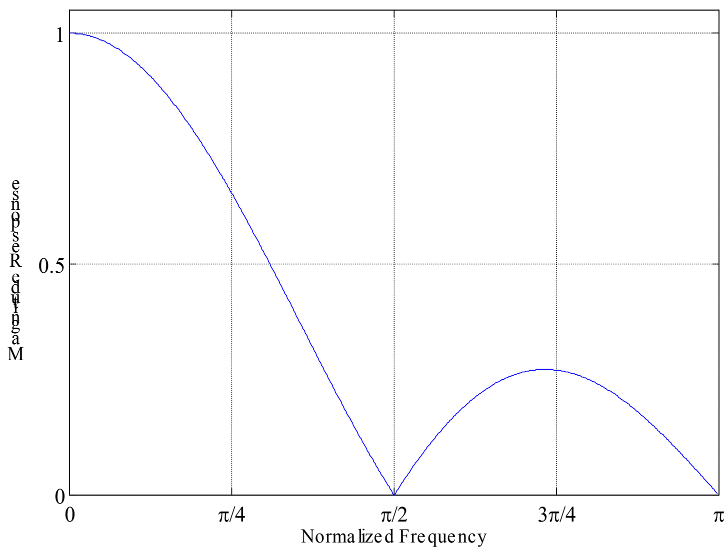
$$= \frac{1}{4} e^{-j\frac{3}{2}\omega} \left(e^{j\frac{3}{2}\omega} + e^{j\frac{1}{2}\omega} 1 + e^{-j\frac{1}{2}\omega} + e^{-j\frac{3}{2}\omega} \right)$$

$$= \frac{1}{2} e^{-j\frac{3}{2}\omega} \left[\cos\frac{3}{2}\omega + \cos\frac{1}{2}\omega \right]$$

Frequency response of 4-point averager

$$H_{lp}(e^{j\omega}) = e^{-j3\omega/2} \cdot \left[\cos\frac{3}{2}\omega + \cos\frac{\omega}{2}\right]$$

Magnitude Response



Phase Response